

Rostocker Zentrum zur Erforschung des Demografischen Wandels
Rostock Center for the Study of Demographic Change
Konrad-Zuse-Strasse 1 · D-18057 Rostock · Germany
Tel.: + 49 (0) 381 2081 – 0 · Fax: +49 (0) 381 2081 – 202
www.rostockerzentrum.de

ROSTOCKER ZENTRUM – DISKUSSIONSPAPIER
ROSTOCK CENTER – DISCUSSION PAPER

No. 20

On the Value of Life and the Value of Progeny

Michael Kuhn
Stefan Wrzaczek
Jim Oeppen

July 2008

On the Value of Life and the Value of Progeny

Michael Kuhn

Max-Planck-Institute for Demographic Research
kuhn@demogr.mpg.de

Stefan Wrzaczek

Vienna University of Technology
wrzaczek@server.eos.tuwien.ac.at

Jim Oeppen

Max-Planck-Institute for Demographic Research
oeppen@demogr.mpg.de

ROSTOCKER ZENTRUM – DISKUSSIONSPAPIER
ROSTOCK CENTER – DISCUSSION PAPER

No. 20

Juli 2008

Rostocker Zentrum zur Erforschung des Demografischen Wandels

Rostock Center for the Study of Demographic Change

Konrad-Zuse-Strasse 1 · D-18057 Rostock · Germany
Tel.: + 49 (0) 381 2081 – 0 · Fax: +49 (0) 381 2081 – 202
www.rostockerzentrum.de

Accepted by the 'editorial board'*

Any opinions expressed here are those of the author(s) and do not necessarily reflect those of the Institute. The Discussion Papers often represent preliminary work and are circulated to encourage discussion. Citation of such a paper should account for its provisional character. A revised version may be available directly from the author.

The Rostock Center for the Study of Demographic Change is a local and virtual research center and a place of communication between science and politics. The center is associated with the University of Rostock and the Max Planck Institute for Demographic Research

*Members of the 'editorial board': Laura Bernardi, Gabriele Doblhammer, Michaela Kreyenfeld, Michael Kuhn, Marc Luy, Claudia Neu, Gerda Neyer, Steffen Schoon, Carsten Ochsen, Rembrandt Scholz, James W. Vaupel

On the Value of Life and the Value of Progeny^{*†‡}

Michael Kuhn,[§] Stefan Wrzaczek,[¶] Jim Oeppen^{||}

March 27, 2008

Abstract

This note generalises the value of life (VOL) to include the preferences for descendants of a dynastically altruistic decision-maker. The VOL can then be decomposed into the value of own survival and the value of progeny. Conventional models tend to underestimate the value of life by measuring the value of own survival alone. We show how the value of progeny depends on the nature of dynastic altruism: Preferences about population/family size vary between the Benthamite (total utility aggregated over all individuals) and Millian (per capita utility of a representative of each cohort) extremes. The lower the value the decision-maker attaches to population size the lower the value of progeny. Within a stationary economy where all individuals (present and future) have the same scope for consumption, for all preferences but the Benthamite extreme, the progenitor attaches a lower value to the life of a (newborn) individual than the individual would itself. We illustrate by way of a numerical example.

1 Introduction

What is the value to society or to a family of an additional individual? What is the willingness to pay for a small increase in own survivorship? These two questions are strongly related yet have been dealt with so far in rather distinct strands of the literature. The first

*This research was financed by the Austrian Science Fund under contract number P18161-N13 ("Control of heterogenous systems").

†The first two authors contributed equal parts to the paper.

‡The authors are grateful for comments by two anonymous referees.

§Max-Planck-Institute for Demographic Research and University of Rostock, MPIDR, Konrad-Zuse-Str. 1, 18057 Rostock, Germany (kuhn@demogr.mpg.de).

¶Institute of Mathematical Methods in Economics (research group on Operations Research and Non-linear Dynamical Systems), Vienna University of Technology, Argentinierstrasse 8, 1040 Vienna, Austria (wrzaczek@server.eos.tuwien.ac.at).

||Max-Planck-Institute for Demographic Research, Konrad-Zuse-Str. 1, 18057 Rostock, Germany (oeppen@demogr.mpg.de).

question has been addressed, explicitly or implicitly, by a macro economic literature on economic growth and population size (e.g. Barro and Becker 1989, Canton and Meijdam 1997 or Arrow et al. 2003). In this literature, a dynastically-altruistic decision maker maximizes an inter-temporal utility function which aggregates the stream of individual utilities arising for successive generations. One issue relates to the implications of different preferences over population size. These can vary between the Benthamite view that the stream of individual utilities should be aggregated across all living individuals and the Millian view that only the average utility (per capita) for each generation should be counted. Canton and Meijdam (1997) show how steady-state and transitory dynamics for an economy depend on these preferences. While the value of an additional individual can be derived within such macro-economic models (Arrow et al. 2003), life-cycle aspects of individual survival are typically ignored.

This latter issue has been addressed by the micro-economic oriented life-cycle literature (e.g. Grossman 1972, Shepard and Zeckhauser 1984, Ehrlich and Chuma 1990, Ehrlich 2000). These models allow the derivation of an age-profile for the value of life from the individual's perspective but typically ignore (altruistic) preferences over progeny or include them only indirectly by assuming that individuals attach a value to leaving bequests. As we will show this may lead to biased (generally too low) values of own life.

In this work we seek to combine the two issues, population and individual survival, and derive a unified value of life that includes the value an individual attaches to its prospective descendants. In so doing we allow for different preferences over the size of the dynasty. This analysis allows us to integrate life-cycle models (enriched by preferences over progeny) and population based models (enriched by a life-cycle part).¹

Our analysis shows how the value of an individual life can be decomposed into the willingness to pay for a small reduction in mortality (the conventional value of life) and the expected value attached to future descendants. It depends on the age-profile of individual fertility (assumed to be exogenous) and on the decision-maker's preferences. We find that the generalized value of life exceeds the conventional value of life (ignoring bequests) under two circumstances: (i) the individual is still in its fertile years and (ii) the individual has Benthamite rather than Millian preferences, i.e. it tends to value the size of the dynasty.

The remainder of the note is organized as follows. In section 2 we present our age-structured model, develop our generalized value of life and discuss its components. Section 3 illustrates by way of a numerical simulation. Section 4 concludes.

2 The Model

Consider a dynastically altruistic decision-maker. This could be the 'head' of a family or it could be a social planner considering the well-being of a population. The objective

¹Hall and Jones (2007) consider age-structure within a population based model to analyse the development of aggregate health expenditure. However, by assuming age-independent fertility and by ignoring the preferences over populations, their model does not allow the generalisation of the value of life undertaken in the present note.

function is defined as

$$\int_0^T \int_0^\omega e^{-\rho t} u(c(a, t)) M(a, t) B^\epsilon(t - a) da dt \quad (1)$$

where $u(c(a, t))$ is the instantaneous utility from consumption $c(a, t)$ for an individual aged a at time t . Furthermore, we denote by $M(a, t)$ the survival probability of an a -year old individual and by $B(t - a)$ the size of the cohort born at $t - a$.² We follow Barro and Becker (1989) and Canton and Meijdam (1997) by allowing variable preferences over population size. These preferences are measured by the parameter $\epsilon \in [0, 1]$. If $\epsilon = 1$, utility is weighted by the size of the cohort, corresponding to Benthamite preferences. If $\epsilon = 0$ only the individual life-cycle utilities are summed up irrespective of cohort size, which is equivalent to a Millian utility function. All other values of ϵ correspond to intermediate cases. Finally ρ denotes the rate of time preference.

The variables of our model evolve in age a and time t , i.e. in two dimensions. The number of individuals in age-group a at time t follows as the product of the survival function at (a, t) and the original cohort size, $N(a, t) = M(a, t)B(t - a)$.³ We can then express the dynamics of the model by the McKendrick partial differential equation (see Keyfitz et al. 1997)

$$N_a + N_t = -\mu(a, h(a, t))N(a, t) \quad N(0, t) = B(t), N(a, 0) = N_0 \quad (2)$$

where $\mu(a, h(a, t))$ denotes the mortality rate for age a at time t depending negatively on period health expenditure $h(a, t)$. The equation describes the change in the population for a small (infinitesimal) unit of time, i.e. from $N(a, t)$ to $N(a + dt, t + dt)$ (following the 45 degree line in the Lexis diagram). The change equals the number of deaths in this interval, described by the mortality rate (for this age and time) times the number of individuals who are still alive. $N_0(a)$ denotes the initial age distribution of the population and $B(t)$ the number of newborns at time t . Here, $B(t) = \int_0^\omega \nu(a)N(a, t)da$ is the aggregate number of births across all age-groups, where $\nu(a)$ denotes the exogenous birth rate of an individual aged a .

As a budget constraint we assume that the assets held by every cohort are balanced. This is modelled by introducing cohort assets $A(a, t)$ with the following dynamics

$$\begin{aligned} A_a + A_t &= rA(a, t) + (y(a) - c(a, t) - h(a, t))N(a, t) \\ A(0, t) &= A(\omega, t) = 0 \quad \forall t \\ A(a, 0) &= A_0(a), A(a, T) = A_T(a) \quad \forall a \end{aligned} \quad (3)$$

²The objective can be understood as the aggregation over time of the life-cycle utilities of the members of subsequent cohorts. Here, $\int_0^\omega e^{-\rho s} u(c(s, t - a + s))M(s, t - a + s) ds$, represents the expected life-time utility of an individual born at time $t - a$.

³Due to simplicity we assume homogeneity within the population. Heterogeneity (e.g. gender, wealth or income, education) could be included by introducing additional state variables. However, this would not change our core results.

where $y(a)$ and r denote an age-specific (exogenous) income and the interest rate, respectively.

We maximize the intertemporal utility (1) with respect to $c(a, t)$ and $h(a, t)$ subject to (2) and (3). For both $c(a, t)$ and $h(a, t)$ we assume Inada conditions, such that non-negativity constraints need not be included explicitly. The solution to the model together with a transformation in order to apply age-specific optimal control theory (see Feichtinger et al. 2003) is presented in the appendix.

From the first-order conditions (reported in the appendix) it follows that the cohorts smooth their consumption over the life-cycle if and only if $r = \rho$. If $\rho > r$ ($\rho < r$) consumption decreases (increases) over the life-cycle. Health expenditure is chosen such that the willingness to pay for a marginal reduction in mortality $\psi^S(a, t)$ equals the health expenditure necessary to attain it, i.e. such that $\psi^S(a, t) = \frac{-1}{\mu_h(a, h(a, t))}$. The required expenditure is the lower the greater (in absolute terms) the impact of health expenditure on mortality, $\mu_h(a, h(a, t))$.

Here $\psi^S(a, t)$, understood as a generalized value of life (VOL), can be written as (for the derivation see the appendix)

$$\begin{aligned} \psi^S(a, t) = & \int_a^\omega \left(\frac{u(c)}{u_c(c)} + y - c - h \right) e^{-r(s-a) - \int_a^s \mu(s', h) ds'} ds \\ & + \int_a^\omega \psi^N(a, t, s) \nu(s) e^{-r(s-a) - \int_a^s \mu(s', h) ds'} ds. \end{aligned} \quad (4)$$

and falls into two distinct components: The first part (RHS of the first line) corresponds to the 'conventional' VOL as derived e.g. by Shepard and Zeckhauser (1984) or Rosen (1988). The second part gives the 'value of progeny', where $\psi^N(a, t, s)$ (with $s \geq a$) denotes the value of a birth at some future time $t - a + s$ for a decision maker of current age a at current time t .⁴ We now discuss the two components in turn.

The conventional VOL gives the value the individual attaches to its own life, given a perfect market for annuities. It comprises the discounted stream of gross consumer surplus $\frac{u(c)}{u_c(c)}$ and net wealth accrual $y - c - h$ over the expected remaining life-span. Evidently the conventional VOL is unrelated to the form of inter-generational altruism (as measured by ϵ). Indeed, it pertains for any individual, egoistic and altruistic alike.

For dynastically altruistic individuals, however, there is a second term, the value of progeny.⁵ This gives the (discounted) value to the decision-maker of direct descendants born at rate $\nu(\cdot)$ over the decision-maker's remaining life-span. The value to an individual

⁴Thus, $t - a + s$ gives the time index corresponding to the decision maker's progressing age s , with $s \geq a$.

⁵Interestingly the value of progeny is analogous to the reproductive value, which is well known in demography and has been firstly introduced by Fisher (1930) (for details we refer to Keyfitz (1977)). In the standard notation of the reproductive value the value of a newborn ψ^N is missing. This is because the reproductive value of demography is measured in terms of individuals (thus ψ^N would equal 1), whereas we measure the additional value of such an individual to the decision maker.

aged a at time t of a newborn born at time $t - a + s$ is given by (for the derivation see the appendix):⁶

$$\begin{aligned} \psi^N(a, t, s) = & \left(\frac{B(t - a + s)}{B(t - a)} \right)^{\epsilon - 1} \frac{u_c(c(0, t - a + s))}{u_c(c(s, t - a + s))} \int_0^\omega \left[\epsilon \frac{u(c(s', t - a + s + s'))}{u_c(c(s', t - a + s + s'))} \right. \\ & \left. + \psi^N(0, t - a + s, s') \nu(s') \right] e^{-rs' - \int_0^{s'} \mu(s'', h) ds''} ds' \end{aligned} \quad (5)$$

The two integral terms give in turn the stream over the expected life course of weighted gross surplus from consumption and of the expected value to the newborn of its own direct descendants. Note that the value of progeny is independent of any wealth effects. This is because we assume zero net wealth accrual over the individual's expected life-time.⁷ The factor $\frac{u_c(c(0, t - a + s))}{u_c(c(s, t - a + s))}$ converts the valuation on the part of the newborn cohort into the decision-maker's current valuation.

The factor $\left(\frac{B(t - a + s)}{B(t - a)} \right)^{\epsilon - 1}$ adjusts for changes in population size. A one-off increase (decrease) in the size of birth-cohort $t - a + s$ within an otherwise stationary population, such that $\frac{B(t - a + s)}{B(t - a)} > (<) 1$, tends to depress (inflate) the value of progeny $\psi^N(a, t, s)$. As one would expect, all but pure Benthamite altruists (for whom $\epsilon = 1$) attach relatively lower (higher) value to individuals of larger (smaller) populations. Further effects arise within a non-stationary population through changes in (i) the value of future descendants $\psi^N(0, t - a + s, s')$, and (ii) through the underlying changes in the fertility and/or mortality rates. An ongoing increase in population, for instance would drive down $\psi^N(0, t - a + s, s')$. However, an offsetting effect arises through the corresponding increase in fertility rates and/or decrease in mortality rates.

Assuming an arbitrarily short interval of zero fertility after birth, the following is now readily verified (a proof is provided in the Appendix).

Proposition 1 *Let $\nu(s) = 0$ for $s < \underline{s}$. Then (i) $\psi^N(a, t, a) = 0$ for $\epsilon = 0$ (ii) $\psi^N(a, t, a) \in \left(0, \psi^S(0, t) \frac{u_c(c(0, t))}{u_c(c(a, t))} \left(\frac{B(t - a)}{B(t)} \right)^{1 - \epsilon} \right)$ for $\epsilon \in (0, 1)$ (iii) $\psi^N(a, t, a) = \psi^S(0, t) \frac{u_c(c(0, t))}{u_c(c(a, t))}$ for $\epsilon = 1$.*

The Proposition provides us with a general result about the comparison between the value $\psi^N(a, t, a)$ assigned to a newborn by a decision-maker aged a at the time of birth t and the value $\psi^S(0, t)$ the newborn would attach to itself if it were to take on the role of decision-maker. Generally, we find that pure Millian decision-makers for whom $\epsilon = 0$ assign a zero value to progeny, implying that they cannot be distinguished in VOL terms from pure individualists. A zero value of progeny follows, surprisingly perhaps, as Millian altruists merely care for a continuation of the family line (by one individual) but not for the size of the dynasty. Under exogenous fertility the continuation of the family is guaranteed (at least in expected terms) and by assumption there is no mechanism for redistribution

⁶The expression follows when evaluating $\xi^N(0, t - a + s)$ as in (14) and substituting into the definition $\psi^N(a, t, s) := \frac{B^{1 - \epsilon}(t - a) \xi^N(0, t - a + s)}{u_c(c(s, t - a + s))}$ (see appendix).

⁷Recall our assumption $A(0, t) = A(\omega, t) = 0$.

across cohorts. In this situation, a Millian altruist is indifferent about progeny and is not willing to forego own consumption for the purpose of extending the family (e.g. by spending extra amounts on health care in order to ensure survival for the purpose of child-bearing). A Millian altruist is not indifferent, however, to different levels of consumption across cohorts. In a more general setting, where resources can be transferred across cohorts, a Millian altruist would place an additional value on survival if this allowed her to transfer extra resources to a generation with lower scope for consumption.

All dynastic preferences involving $\epsilon > 0$ imply a positive value of progeny. The generalized VOL for these individuals therefore exceeds the conventional VOL and, thus, the VOL corresponding to egoistic individuals. To obtain a second-result in a clear-cut way we assume a steady-state with perfect consumption smoothing, such that $c(0, t) = c(a, t)$ and $B(t - a) = B(t)$.

Corollary 1 *Let the assumptions of the above Proposition hold in a steady state. Then (i) $\psi^N(a, t, a) \in (0, \psi^S(0, t))$ for $\epsilon \in (0, 1)$ (ii) $\psi^N(a, t, a) = \psi^S(0, t)$ for $\epsilon = 1$.*

For all types of dynastic altruism other than the pure Benthamite case, i.e. for all $\epsilon < 1$, the value of progeny $\psi^N(a, t, a)$ (from the viewpoint of the decision-maker) falls short of the VOL, $\psi^S(0, t)$, the newborn would attach to itself. This is a strong result in as far as it applies irrespective of any potential costs of rearing children. All but perfectly Benthamite altruists are only to a limited extent prepared to forego own consumption for the purpose of extending the family. Solely Benthamite altruists ($\epsilon = 1$) internalize fully the (discounted) VOL of their descendants. We have argued already that population change or changes over cohorts in consumption levels imply adjustments in the value of progeny. In this case the value of progeny can no longer be compared directly with the VOL of a newborn. The basis of reference is now an adjusted VOL, as in parts (ii) and (iii) of the Proposition.

3 Numerical Results

For the numerical calculation we used the following functional specification

$$\begin{aligned} u(c(a, t)) &= b + \frac{c(a, t)^{1-\sigma}}{1-\sigma} \\ \mu(a, h(a, t)) &= \tilde{\mu}(a) \left(1 - \sqrt{\frac{h(a, t)}{z} \left(a \frac{1-d}{1-\omega} + \frac{d-\omega}{1-\omega} \right)} \right) \end{aligned} \quad (6)$$

where $b = 5$, $\sigma = 2$ and $z = 3$ and $d = 0$. We assume an equal time discount and market interest rate $\rho = r = 0.03$. The maximal life-span ω is set to 110. $\tilde{\mu}(a)$ is the age-specific base mortality rate, which is effective when nothing is expended on health.⁸ We proxy individual income by the age-specific marginal product of labour (=wages). The age-specific productivity profile has been estimated by Skirbekk (2005) as a weighted average

⁸For a more detailed discussion of the functional specification we refer to Kuhn et al. (2007).

over 6 age-dependent activities. For the mortality data we used the human mortality data base for the years 1990-2000. For the fertility rates we used the data base of the United Nations Population Division for the years 1995-2000.

Figure 1 plots the VOL for three different intertemporal utility functions: Benthamite ($\epsilon = 1$), Millian ($\epsilon = 0$), and an intermediate specification ($\epsilon = 0.5$). Obviously the Benthamite case (right panel) is associated with the highest VOL since the (discounted) VOL of all future descendants is fully taken into account. Thus a reduction of the mortality rate (at (a, t)) increases the decision-maker's objective not only by the value of higher own survival but also because of a greater number of expected descendants. The Millian case (left panel) shows the lowest VOL due to the fact that no value is assigned to progeny. The VOL for the intermediate case (middle panel) with $\epsilon = 0.5$ lies between the two polar cases. The VOL for the Benthamite case and for the intermediate case increases up to age ≈ 20 , decreases rapidly up to age ≈ 40 , from which age it is no longer distinguishable from the conventional VOL. This shape relates, of course, to the age-distribution of fertility. Thus, while the value of progeny in of itself is still positive even at higher ages, fertility rates close or equal to zero eliminate its impact on the VOL.

It is instructive to compare the age-profile of the generalized VOL with the age-profile of the VOL presented in Shepard and Zeckhauser (1984, page 434, Figure 3) for their 'perfect markets' case. In their model the decision-maker ignores future descendants and thus the age-profile corresponds to that of the conventional VOL in our model. Only for our Millian case, where the value of progeny is zero, does the VOL as a whole show a very similar shape.

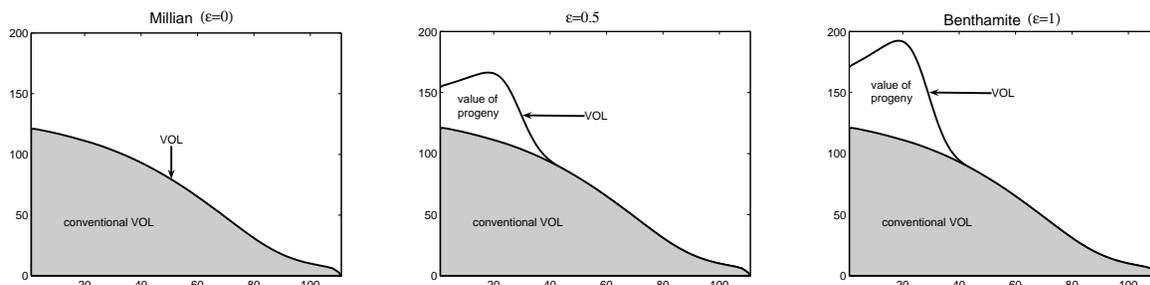


Figure 1: The VOL, conventional VOL and the value of progeny

4 Conclusion

It was the aim of this note to assess the role of dynastic altruism (towards own children) for individual life-cycle behaviour as, indeed, for the value of life (VOL). We show that all but strictly Millian altruists exhibit a VOL in excess of the pure individualists. This is because they include a positive value of progeny. However, all but strictly Benthamite altruists attach a value to their offspring below the VOL at the point of birth. This is

because all but Benthamite altruists are unwilling to trade-off consumption against having a greater number of children. Indeed, Millian altruists who do not care at all for the size of dynasty are not attaching any value to progeny.

Our analysis suggests that studies seeking to establish VOL measures from individual earnings data (e.g. Shepard and Zeckhauser 1984) may lead to underestimates if individuals have altruistic feelings towards their descendants. However, it is also clear that underestimation is only significant for those individuals who are still well within their fertile years. For older individuals the generalized VOL approaches the conventional VOL. It is not the purpose of this note to assess the potential policy-implications of biases in VOL measures.⁹ Broadly speaking, however, our results suggest that earnings-based VOL measures may lead to biases against the valuation of young lives in public safety programmes. Within the provision of health care, this bias appears less problematic for the industrial countries, where mortality reductions assume a significant role only in the post-reproductive stages of life. However, the issue may well be of relevance when thinking of countries such as developing or transition countries which exhibit significant mortality within younger age-groups. This may relate to African countries where mortality is high due to HIV or Post-Soviet countries such as Russia, exhibiting high mortality in particular among younger men.

In principle, feelings - altruistic or other - towards future generations can be modelled in other ways. For instance, bequest preferences can be included in the individual's utility (e.g. Ehrlich 2000). In the presence of perfect annuity markets bequest preferences tend to reduce the net value of survival (Ehrlich 2000), as the value of leaving a bequest now imposes an opportunity cost. This form of inter-generational altruism therefore leads to a result which is opposite to our findings.¹⁰ Alternatively, the decision-maker may enjoy an (emotional) per-period benefit of having descendants around.¹¹ This would raise the value of life not only over the fecund period of life but also at higher ages.

We have hinted already at the interest of extending the model to allow for inter-generational redistribution. Scope for transfers across cohorts arises if wealth is distributed unequally across cohorts or if certain age-groups face tighter credit constraints than others. In this case, dynastically minded altruists will have an interest (potentially) to make inter-generational transfers in order to redistribute consumption across cohorts.¹² Our model suggests that besides a shift in the age-profile of VOL, wealth effects may then have a bearing on the value of progeny. Finally, the model may well lend itself towards an analy-

⁹See Ashenfelter (2006) for a recent survey on the application of VOL measures.

¹⁰The introduction of bequest preferences in Ehrlich (2000) leaves the issues unresolved as to (i) why individuals prefer to transfer resources to descendants in form of bequests rather than inter-vivos and (ii) how the benefit from bequeathing relates to the number of offspring.

¹¹See e.g. Basu and Meltzer (2005) who present a life-cycle model where the members of a two-person household (i.e. husband and wife) care for each other so that their own and their partner's survival become complementary in generating life-cycle benefits. Basu and Meltzer (2005) do not, however, incorporate descendants into their model.

¹²When discussing the appropriate age-weighting for life-years lost due to disease Murray and Acharya (1997) point at evidence suggesting that people place a greater weight on life-years during which individuals assume an important role in supporting dependents.

sis of fertility choices while taking into account the cost of raising children. As all of these analyses call for non-trivial extensions of the model we defer them to future research.

A Appendix

Optimization: The formulation in the main text does not allow the application of the necessary optimality conditions (see Feichtinger et al. 2003), as the time argument in the intertemporal utility function is delayed. Whereas c and N are valued at time t , B is valued at the cohorts birth time $t - a$. However, this can be avoided by including survival probability as state with the dynamics

$$M_a + M_t = -\mu(a, h(a, t))M(a, t) \quad M(a, 0) = M_0(a), M(0, t) = 1 \quad (7)$$

where $M(a, t)$ denotes the survival probability of an individual born at time $t - a$ to age a (reached at time t). Substituting $B(t - a) = \frac{N(a, t)}{M(a, t)}$ yields

$$\int_0^T \int_0^\omega e^{-\rho t} u(c(a, t)) M^{1-\epsilon}(a, t) N^\epsilon(a, t) da dt \quad (8)$$

for (1).

The Hamiltonian of the optimization problem reads as follows (from now on we skip a and t if they are not of particular importance)

$$\mathcal{H} = u(c)M^{1-\epsilon}N^\epsilon - \xi^N \mu(a, h)N - \xi^M \mu(a, h)M + \xi^A(rA + (y - c - h)N) + \eta^B \nu N \quad (9)$$

where ξ^N , ξ^M , ξ^A and η^B denote the adjoint variables relating to the states N , M , A , and to the integral state B , respectively. Applying age-specific control theory we obtain the following adjoint system

$$\begin{aligned} \xi_a^N + \xi_t^N &= (\rho + \mu(a, h))\xi^N - \epsilon u(c) \left(\frac{N}{M}\right)^{\epsilon-1} - \xi^A(y - c - h) - \eta^B \nu \\ \xi_a^M + \xi_t^M &= (\rho + \mu(a, h))\xi^M - (1 - \epsilon)u(c) \left(\frac{N}{M}\right)^\epsilon \\ \xi_a^A + \xi_t^A &= (\rho - r)\xi^A \\ \eta^B &= \xi^N(0, t) \end{aligned} \quad (10)$$

together with the transversality conditions $\xi^N(a, T) = \xi^N(\omega, t) = 0$ and $\xi^M(a, T) = \xi^M(\omega, t) = 0$. Since initial and end constraints with respect to cohort assets cannot formally be included in the model for all the cohorts, we add the terms $-\lambda \int_0^T e^{-rt} A^2(\omega, t) dt$ and $-\lambda \int_0^\omega e^{-rT} (A(a, T) - A_T(a))^2 da$ to the objective functions, yielding $\xi^A(\omega, t) = -2\lambda A(\omega, t)$ and $\xi^A(a, T) = -2\lambda(A(a, T) - A_T(a))$ as transversality conditions.

Finally, we can derive the necessary first order conditions for c and h

$$\mathcal{H}_c = u_c(c)M^{1-\epsilon}N^\epsilon - \xi^A N = 0 \quad (11)$$

$$\mathcal{H}_h = -\xi^N \mu_h(a, h)N - \xi^M \mu_h(a, h)M - \xi^A N = 0 \quad (12)$$

Derivation of the value of life (VOL): Following, e.g. Rosen (1988), we obtain the following expression for the value of an individual life

$$\psi^S(a, t) = -\frac{1}{N} \frac{\partial V / \partial \mu}{\partial V / \partial A} = \frac{1}{u_c(c(a, t))} \left(\xi^N(a, t) B^{1-\epsilon}(t-a) + \xi^M(a, t) B^{-\epsilon}(t-a) \right) \quad (13)$$

where V denotes the value function, and where $\frac{\partial V}{\partial \mu} = \frac{\partial V}{\partial N} \frac{\partial N}{\partial \mu} + \frac{\partial V}{\partial M} \frac{\partial M}{\partial \mu} = -\xi^N N - \xi^M M = -(\xi^N + \xi^M B^{-1}(t-a))N$ and $\frac{\partial V}{\partial A} = \xi^A = u_c(c(a, t))B^{\epsilon-1}(t-a)$, respectively.¹³ The shadow price $\xi^N(a, t)$, which can be interpreted as the increase in the objective if there is one more individual aged a at time t , can be calculated from the adjoint equation with the method of characteristics

$$\xi^N(a, t) = \int_a^\omega \left((\epsilon u(c) + u_c(c)(y - c - h)) B^{\epsilon-1}(t-a) + \xi^N(0, t-a+s)\nu(s) \right) \times e^{-\rho(s-a) - \int_a^s \mu(s', h) ds'} ds \quad (14)$$

Likewise, we obtain

$$\xi^M(a, t) = \int_a^\omega (1 - \epsilon)u(c)B^\epsilon(t-a)e^{-\rho(s-a) - \int_a^s \mu(s', h) ds'} ds \quad (15)$$

Substituting from (14) and (15) into (13), substituting

$$u_c(c(a, t)) = \frac{\xi^A(a, t)}{\xi^A(s, t-a+s)} u_c(c(s, t-a+s)) = u_c(c(s, t-a+s))e^{-(\rho-r)(s-a)} \quad (16)$$

collecting terms, and using the definition $\psi^N(a, t, s) := \frac{B^{1-\epsilon}(t-a)\xi^N(0, t-a+s)}{u_c(c(s, t-a+s))}$ we obtain the expression reported in (4).

Derivation of the value of progeny: Evaluating the expression in (14) at $(0, t-a+s)$ and substituting into $\psi^N(a, t, s) := \frac{B^{1-\epsilon}(t-a)\xi^N(0, t-a+s)}{u_c(c(s, t-a+s))}$ we obtain

$$\begin{aligned} \psi^N(a, t, s) = & \frac{B^{1-\epsilon}(t-a)}{u_c(c(s, t-a+s))} \int_0^\omega \left[\left(\epsilon u(c(s', t-a+s+s')) \right) \right. \\ & \left. + u_c(c(s', t-a+s+s'))z(s', t-a+s+s') \right) B^{\epsilon-1}(t-a+s) + \\ & \left. + \xi^N(0, t-a+s+s')\nu(s') \right] e^{-\rho(s-a) - \int_a^s \mu(s', h) ds'} ds' \end{aligned}$$

¹³Note (i) that division by N in (13) converts the value of $\frac{\partial V / \partial \mu}{\partial V / \partial A}$, which is given at population level, into a per-capita value; and (ii) that $\xi^A = u_c(c(a, t))B^{\epsilon-1}(t-a)$ follows from (11).

with $z(\cdot, \cdot) := y(\cdot, \cdot) - c(\cdot, \cdot) - h(\cdot, \cdot)$. Substituting from $u_c(c(s', t - a + s + s')) = \frac{\xi^A(s', t - a + s + s')}{\xi^A(0, t - a + s)} u_c(c(0, t - a + s)) = u_c(c(0, t - a + s)) e^{(\rho - r)s'}$, observing that

$$\int_0^\omega z(s', t - a + s + s') e^{-r(s-a) - \int_a^s \mu(s', h) ds'} ds' = 0 \quad (17)$$

follows from $A(0, t) = A(\omega, t) = 0$, and rearranging terms gives the expression reported in (5).

Proof of Proposition:

Part (i): $\epsilon = 0$. Since $\psi^N(a, t, a) = \frac{B(t-a)\xi^N(0, t)}{u_c(c(a, t))}$ we have to show that $\xi^N(0, t) = 0$ holds.

Analogously to the derivation above we obtain $\int_0^\omega u_c(c(s, t + s))(y(s) - c(s, t + s) - h(s, t + s)) B^{\epsilon-1}(t) e^{-\rho s - \int_0^s \mu(s', h) ds'} ds = 0$. Thus (14) can be expressed as

$$\begin{aligned} \xi^N(0, t) &= \int_0^\omega \pi(s_1) \nu(s_1) \xi^N(0, t + s_1) ds_1 = \\ &= \int_0^\omega \int_0^\omega \pi(s_1) \pi(s_2) \nu(s_1) \nu(s_2) \xi^N(0, t + s_1 + s_2) ds_1 ds_2 = \\ &= \dots = \\ &= \int_0^\omega \dots \int_0^\omega \pi(s_1) \dots \pi(s_\delta) \nu(s_1) \dots \nu(s_\delta) \xi^N(0, t + s_1 + \dots + s_\delta) ds_1 \dots ds_\delta \end{aligned} \quad (18)$$

where¹⁴ $\pi(s) := \exp(-\rho s - \int_0^s \mu(s', h) ds') > 0$ and $t + \delta \underline{s} < T < t + (\delta + 1) \underline{s}$. Whenever $t + s_1 + \dots + s_\delta < T$. Then we have to distinguish three cases, each leading to a situation where at least one fertility rate is zero. (i.i) If $t + s_1 + \dots + s_\delta < t + \delta \underline{s}$, there exists at least one $s_i \leq \underline{s}$ implying $\nu(s_i) = 0$. (i.ii) If $t + \delta \underline{s} < t + s_1 + \dots + s_\delta < T$ we use (14) again and obtain $\xi^N(0, t + s_1 + \dots + s_\delta) = \int_0^{T - (t + s_1 + \dots + s_\delta)} \pi(s) \nu(s) \xi^N(0, t + s_1 + \dots + s_\delta + s) ds = 0$ since $\nu(s) = 0$ by $s \leq \underline{s}$. (i.iii) If $t + s_1 + \dots + s_\delta > T$ we make the following definition (we can make this definition as $\xi^N(\cdot)$ is not defined outside the planning horizon¹⁵): $\xi^N(0, t + s_1 + \dots + s_\delta) := \frac{1}{\pi(s_i) \dots \pi(s_\delta)} \frac{1}{\nu(s_1) \dots \nu(s_\delta)} \frac{1}{\omega^{\delta - (i-1)}} \xi^N(0, t + s_1 + \dots + s_{i-1})$. Then $t + s_1 + \dots + s_{i-1} < T < t + s_1 + \dots + s_{i-1} + \underline{s}$ and cases (i.i) and (i.ii) can be applied for $\xi^N(0, t + s_1 + \dots + s_{i-1})$.

¹⁴For notational simplicity we omit the time argument in the definition of $\pi(\cdot)$.

¹⁵Note that we make here a step backwards in the recursion. This could have been avoided by making the forward recursion in another way. However, this would have resulted in a very unclear notation. Because of this step backwards we know that the expression inserted in the original integral exists.

Part (ii): $\epsilon \in (0, 1)$. Multiplying (13) by $\frac{u_c(c(0,t))}{u_c(c(a,t))}$ we have

$$\begin{aligned} \psi^S(0,t) \frac{u_c(c(0,t))}{u_c(c(a,t))} \left(\frac{B(t-a)}{B(t)} \right)^{1-\epsilon} &= \frac{B^{1-\epsilon}(t) \xi^N(0,t) u_c(c(0,t))}{u_c(c(0,t)) u_c(c(a,t))} \left(\frac{B(t-a)}{B(t)} \right)^{1-\epsilon} + \\ &+ \frac{B^{-\epsilon}(t) \xi^M(0,t) u_c(c(0,t))}{u_c(c(0,t)) u_c(c(a,t))} \left(\frac{B(t-a)}{B(t)} \right)^{1-\epsilon} = \\ &= \frac{B^{1-\epsilon}(t-a) \xi^N(0,t)}{u_c(c(a,t))} + \frac{\xi^M(0,t) B^{1-\epsilon}(t-a)}{u_c(c(a,t)) B(t)} \end{aligned} \quad (19)$$

The first term equals the definition of $\psi^N(a, t, a)$. (15) implies positivity of the second term. Thus $\psi^N(a, t, a) < \psi^S(0, t) \frac{u_c(c(0,t))}{u_c(c(a,t))}$.

Part (iii): $\epsilon = 1$. Since (15) implies $\xi^M(0, t) = 0$, (13) multiplied by $\frac{u_c(c(0,t))}{u_c(c(a,t))}$ yields

$$\psi^S(0,t) \frac{u_c(c(0,t))}{u_c(c(a,t))} = \frac{\xi^N(0,t) u_c(c(0,t))}{u_c(c(0,t)) u_c(c(a,t))} = \frac{\xi^N(0,t)}{u_c(c(a,t))} = \psi^N(a, t, a) \quad (20)$$

by definition. ■

References

- [1] Ashenfelter O. (2006): *Measuring the Value of a Statistical Life: Problems and Prospects*, IZA Discussion Paper 1911.
- [2] Arrow K., P. Dasgupta and K.-G. Maeler (2003): *The genuine savings criterion and the value of population*, Economic Theory 21, 217-225.
- [3] Barro R. J. and G. S. Becker (1989): *Fertility choice in a model of economic growth*, Econometrica 57, 481-501.
- [4] Basu A. and D. Meltzer (2005): *Implications of spillover effects within the family for medical cost-effectiveness analysis*, Journal of Health Economics 25, 751-773.
- [5] Canton E. and L. Meijdam (1997): *Altruism and the macroeconomic effects of demographic changes*, Journal of Population Economics 10, 317-334.
- [6] Ehrlich I. and H. Chuma (1990): *A model of the demand for longevity and the value of life extension*, Journal of Political Economy 98 (4), 761-782.
- [7] Ehrlich I. (2000): *Uncertain lifetime, life protection, and the value of life saving*, Journal of Health Economics 19, 341-367.
- [8] Feichtinger G., G. Tragler and V. M. Veliov (2003): *Optimality conditions for age-structured control systems*, Journal of Mathematical Analysis and Applications 288, 47-68.

- [9] Fisher R. A. (1930): *The general theory of natural selection*, Dover Publications, New York.
- [10] Grossman M. (1972): *On the concept of health capital and the demand for health*, Journal of Political Economy 80 (2), 223-255.
- [11] Hall R. E. and C. I. Jones (2007): *The value of life and the rise in health spending*, Quarterly Journal of Economics 122, 39-72.
- [12] Keyfitz N. (1997): *Applied mathematical demography*, John Wiley & Sons.
- [13] Keyfitz B. L. and N. Keyfitz (1997): *The McKendrick partial differential equation and its uses in epidemiology and population study*, Mathematical Computation Modelling 26 (6), 1-9.
- [14] Kuhn M., A. Prskawetz, S. Wrzaczek and G. Feichtinger (2007): *Health, Survival and Consumption over the Life Cycle: Individual versus Social Optimum and the Role of Externalities*, Rostock Center-Discussion Paper No. 16.
- [15] Murray C. J. L. and A. K. Acharya (1997): *Understanding DALYs*, Journal of Health Economics 16, 703-730.
- [16] Rosen S. (1988): *The value of changes in life expectancy*, Journal of Risk and Uncertainty 11, 285-304.
- [17] Shepard D. S. and R. J. Zeckhauser (1984): *Survival versus consumption*, Management Science 30, 423-439.
- [18] Skirbekk V. (2005): *Why not start younger? Implications of the timing and duration of schooling for fertility, human capital, productivity, and public pensions*, International Institute for Applied Systems Analysis, Laxenburg, Austria.